Exercise 6

1. Compute the second fundamental form with respect to the unit normal $\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$ and the Gaussian curvature of the following parametrized surfaces:

   (a) $\mathbf{x}(u,v) = (u^2 - v^2, 2uv, u^2 + v^2)$, $(u,v) \in \mathbb{R}^+ \times \mathbb{R}^+$.

   (b) $\mathbf{x}(u,v) = (u, v, uv)$, $(u,v) \in \mathbb{R}^2$

   (c) $\mathbf{x}(u,v) = (u^3 \cos v, u^3 \sin v, u)$, $(u,v) \in \mathbb{R}^+ \times (0, 2\pi)$

2. Suppose the first fundamental form of a regular parametrized surface $\mathbf{x}(u,v)$ is

$$I = \begin{pmatrix} f^2 & 0 \\ 0 & f^2 \end{pmatrix}$$

   where $f = f(u,v) > 0$ is a smooth function.

   (a) Show that

$$K = -\frac{1}{f^2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) \ln f$$

   (b) Hence, compute the Gaussian curvature of the following parametrized surfaces with the following first fundamental forms:

   (i) $I = \begin{pmatrix} \frac{1}{u^2+v^2+1} & 0 \\ 0 & \frac{1}{u^2+v^2+1} \end{pmatrix}$

   (ii) $I = \begin{pmatrix} e^{-u^2} & 0 \\ 0 & e^{-u^2} \end{pmatrix}$

3. Let $\mathbf{x}(u,v)$ be a regular parametrized surface, given by

$$\mathbf{x}(u,v) = (u \cos v, u \sin v, v)\,,\ (u,v) \in (0, +\infty) \times (0, 2\pi)$$

   (a) Compute the Gauss map of $\mathbf{x}$ at the point (u,v).

   (b) Is the surface $\mathbf{x}(u,v)$ an orthogonal parametrization?

   (c) Hence, compute the Gaussian curvature of the surface.

4. Let S be a regular surface in $\mathbb{R}^3$ and H be the mean curvature of S.

   (a) State the condition that S is a *minimal surface*.

   (b) Let Φ denote the surface obtained by rotating the graph of $x = f(z)$ on the xz-plane about the z-axis. Suppose that Φ is a regular surface.

   (i) Show that the mean and Gaussian curvature of the surface Φ are given by

$$H_\Phi(z) = \pm\frac{1 + (f'(z))^2 - f(z)f''(z)}{2f(z)\,(1 + (f'(z))^2)^{\frac{3}{2}}} \quad \text{and} \quad K_\Phi(z) = \frac{-f''(z)}{f(z)\,(1 + (f'(z))^2)^2}$$

   respectively.

   (ii) Using part (a), verify that the surface obtained by rotating the graph of

$$x = a\left(\cosh\frac{z}{a}\cosh b + \sinh\frac{z}{a}\sinh b\right)$$

   with $a > 0$ and $b \in \mathbb{R}$ is a **minimal surface**.

5. Let $\mathbf{X}(u,v)$ be a regular parametrized surface and $\mathbf{n} = \dfrac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|}$ be the unit normal vector. Denote the first and second fundamental forms of $\mathbf{X}(u,v)$ by $\mathbf{I}$ and $\mathbf{II}$ respectively. Let

$$\begin{cases} \mathbf{n}_u = a_{11}\mathbf{x}_u + a_{12}\mathbf{x}_v \\ \mathbf{n}_v = a_{21}\mathbf{x}_u + a_{22}\mathbf{x}_v \end{cases}$$

(a) Show that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -(\mathbf{II})(\mathbf{I})^{-1}.$$

(**Note.** This matrix represents the ***differential of gauss map***)

(b) Define the **shape operator** of $\mathbf{X}$ by the **negative** differential of gauss map. It's matrix representation is denoted by 2×2 matrix S. Show that

$$S = (\mathbf{II})(\mathbf{I})^{-1}.$$

(c) Denote the Gaussian curvature and the mean curvature of $\mathbf{X}$ by K and H. Prove that

$$\begin{cases} K = \det(S) \\ H = \dfrac{1}{2}\mathrm{tr}(S) \end{cases}$$

(d) Let $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ be a 2×2 matrix with $a, b, c \in \mathbb{R}$. Show that

$$A^2 - \mathrm{tr}(A)A + \det(A)I = \mathbf{0} \tag{1}$$

where $\mathbf{0}$ denotes the 2×2 zero matrix.
Hence, using (1) and part (c), show that

$$S^2 - 2HS + KI = \mathbf{0}$$

(e) It is given that the matrix representation of the shape operator of $\mathbf{X}$ at $\mathbf{p} = (u,v)$ is

$$S = (\mathbf{II})(\mathbf{I})^{-1} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

(i) Write down the Gaussian curvature $K(\mathbf{p})$ and the mean curvature $H(\mathbf{p})$ of $\mathbf{X}$ at the point $\mathbf{p}$.

(ii) By considering $\det(S - \kappa I) = 0$, compute the **principal curvatures** of $\mathbf{X}$ at $\mathbf{p}$.

(iii) Hence, using (e)(ii), find the corresponding **principal directions**.

(iv) Let $\kappa_1 \le \kappa_2$ be the principal curvatures of $\mathbf{X}$ at $\mathbf{p}$ which associate with two *orthogonal* principal directions in (e)(iii). For any unit vectors $\mathbf{v} = T_{\mathbf{p}}(\mathbf{X}(u,v))$ tangent to $\mathbf{X}$ at $\mathbf{p}$. Denote the **normal curvature** along the direction $\mathbf{v}$ by $\kappa_n(\mathbf{v})$. Find a range to estimate $\kappa_n(\mathbf{v}$.)